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VARIATIONAL PRINCIPLES AND ESTIMATES FOR THE RIGIDITIES OF BODIES WITH VOIDS

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Variational principles and estimates (including two-sided estimates) are obtained for the rigidities of bodies containing periodic systems of pores (voids). The problem is examined on the basis of the asymptotic averaging method.

Formulation of the Problem. We are examining three-dimensional composites, plates, and beams. By their rigidities, we mean the averaged elastic constants: in-plane and bending rigidities for a three-dimensional composite, tensile and bending rigidities for a plate, and torsional rigidities for a beam. Rigidities are calculated by means of formulas that follow from the asymptotic method in [1]. We note that most of the bodies with periodically positioned voids are plates and beams.

We examine an elastic body with a periodic structure having the periodicity cell P_ε, where ε is the characteristic dimension of the cell. We have a three-dimensional composite when the periodicity cell is repeated along three coordinates, a plate when the periodicity cell is repeated along two coordinates, and a beam when the periodicity cell is repeated along one coordinate. The resulting region is denoted by Q_ε. The elastic constants of the body a_{ijkl} are functions of the argument x/ε and are periodic along x_1, x_2, and x_3 for the composite, along x_1 and x_2 for the plate, and along x_1 for the beam.

We examine the problem of the theory of elasticity in the region Q_ε. It is known [1–5] that as $\varepsilon \to 0$ the solution of this problem approaches the solution of problems of the theory of elasticity for uniform bodies [1–3], plate theory [3, 4], or beam theory [3, 5]. To obtain the rigidity characteristics of the above-indicated limiting bodies, we solve a so-called cellular problem. This problem can be written as follows for all of the cases considered:

$$(a_{ijkl}(\mathrm{y})(N^M_{k,l} + f^M_{kl}(\mathrm{y})))_{,j} = 0 \quad \text{in } P_1; \tag{1.1}$$

$$a_{ijkl}(\mathrm{y})(N^M_{k,l} + f^M_{kl}(\mathrm{y}))n_j = 0 \quad \text{on } \gamma \cup \Gamma; \tag{1.2}$$

$$\mathbf{N}^M(\mathrm{y}) \ \text{ is periodic over } \ y_m \ \ (m \in D). \tag{1.3}$$

Here $\mathrm{y} = \mathrm{x}/\varepsilon$ are dimensionless variables, $P_1 = \varepsilon^{-1}P_\varepsilon = \{\mathrm{y} = \mathrm{x}/\varepsilon : \mathrm{x} \in P_\varepsilon\}$ is a periodicity cell in dimensionless variables, $\mathbf{n}$ is a normal to $\gamma \cup \Gamma$ (Fig. 1), M is a multiple index ($D = \{1, 2, 3\}$ for three-dimensional composites, $D = \{1, 2\}$ for plates, and $D = \{1\}$ for beams).

The rigidity characteristics are calculated from the formula

$$A^M = \langle a_{ijkl}(\mathrm{y})(N^M_{i,j} + f^M_{ij}(\mathrm{y}))(N^M_{k,l} + f^M_{kl}(\mathrm{y}))\rangle, \tag{1.4}$$

where $\langle\ \rangle = \dfrac{1}{\operatorname{mes} S}\displaystyle\int_{P_1} dy$ is the mean over a periodicity cell in the dimensionless coordinates y. Here $S = P_1 \cup Q_1$ for three-dimensional composites, S is the projection of $P_1 \cup Q_1$ onto the plane Oy_1y_2 for plates and onto the Oy_1 axis for beams.

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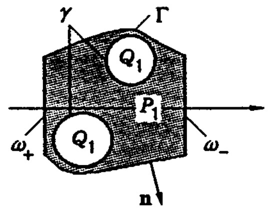

Fig. 1 Periodicity cell voids P_1 for a beam: Q_1 are the, Γ is the free surface, γ are the boundaries of voids, ω_+ and ω_- are the adjacent boundaries of the periodicity cell.

Formula (1.2) was derived in [1, 2] for three-dimensional composites, in [4] for plates of constant thickness, and in [6] for cylindrical beams. In the general case, they are derived from the cellular problem (1.1), as they were in the studies just cited. The formulas for the averaged characteristics [1–6] have the form

$$A^M = \langle(-1)^\mu y^\mu (a_{ijkl}(\mathrm{y})(-1)^\nu y^\nu + a_{ijmn}(\mathrm{y}) N^{kl\nu}_{m,n}(\mathrm{y}))\rangle \tag{1.5}$$

$[M = (ijkl\nu\mu)$, values of the indices having been given in [1–6] and thus being omitted here].

We use formula (1.2) as the initial formula. The elastic constants a_{ijkl} are subject to the standard conditions [7]: for all $\mathrm{y} \in P_1$, we have

$$|a_{ijkl}(\mathrm{y})| \leqslant C < \infty; \tag{1.6}$$

$$a_{ijkl}(\mathrm{y}) e_{ij} e_{kl} \geqslant c|e_{ij}|^2 \quad \text{for any} \quad e_{ij} = e_{ji}, \tag{1.7}$$

where $0 < c$ and $C < \infty$ are independent of y and e_{ij}.

2. Variational Properties and Estimates for Rigidity. For the cellular problem (1.1)–(1.3) we introduce the Lagrange functional

$$J_u(\mathrm{u}) = G(\Lambda \mathrm{u}), \tag{2.1}$$

where

$$G(p) = \frac{1}{2}\langle -a_{ijkl}(\mathrm{y}) p_{ij} p_{kl} - 2a_{ijkl}(\mathrm{y}) f^M_{ij}(\mathrm{y}) p_{kl}\rangle; \tag{2.2}$$

$$(\Lambda \mathrm{u})_{ij} = u_{i,j}. \tag{2.3}$$

The Lagrange functional is examined on the set of possible displacements:

$$V = \{\mathrm{u} \in H^1(P_1)\colon \mathrm{u}(\mathrm{y}) \text{ is periodic over } y_m \ (m \in D)\}. \tag{2.4}$$

Equations (1.1)–(1.3) are the equations of the Euler variational problem

$$J_u(\mathrm{u}) \to \max, \quad \mathrm{u} \in V. \tag{2.5}$$

If conditions (1.6)–(1.7) are satisfied, the solutions of problems (1.1)–(1.3) are accurate to within the displacement values corresponding to the displacements of a rigid body [1–5]. On the set $V_0 = \{\mathrm{u} \in V\colon \langle \mathrm{u}\rangle = 0\}$ ($\langle \mathrm{u}_\gamma S_\gamma \mathrm{y}_{\bar\gamma}\rangle = 0$ for a beam), the solutions of these problems are unique and coincide, since the equality

$$A^M = \langle a_{ijkl}(\mathrm{y}) f^M_{ij}(\mathrm{y}) f^M_{kl}(\mathrm{y})\rangle - 2J_u(\mathrm{N}^M) \tag{2.6}$$

follows from (1.1), (1.4), and (2.1). Here $J_u(\mathrm{N}^M) = \max_{\mathrm{u} \in V} J_u(\mathrm{u})$.

Now we shall obtain the Castigliano functional. Following [8], we introduce the space $H = \{L_2(P_1)\}^6$ and its dual conjugate space H^*. The dual problem of (2.5) is the problem [8]

$$h^*(p^*) \to \min, \quad p^* \in H^*, \tag{2.7}$$

where

$$h^*(p^*) = \sup_{\mathrm{u} \in V} \sup_{q \in H} [\langle \Lambda^* p^*, \mathrm{u}\rangle - \langle p^*, q\rangle - G(q)] = \sup_{\mathrm{u} \in V} [\langle \Lambda^* p^*, \mathrm{u}\rangle - G^*(p^*)]. \tag{2.8}$$

Here $G^*(p^*)$ is a functional conjugate with $G(p)$, $\Lambda^*: H^* \to H$ is an operator conjugate with Λ, and $\langle\, ,\, \rangle$ denotes the operation of pairing of the elements of V^* and V.

We examine the term $\langle \Lambda^* p^*, \mathbf{u}\rangle$. It can be written in the form

$$\langle p^*, \Lambda \mathbf{u}\rangle = \frac{1}{\operatorname{mes} S} \int_{P_1} p^*_{ij} u_{i,j}\, dy = \frac{1}{\operatorname{mes} S}\Big[-\int_{P_1} p^*_{ij,j} u_i\, dy + \int_{\gamma\cup\Gamma} p^*_{ij} n_j u_i\, dy + \int_{\omega} p^*_{ij} n_j u_i\, dy\Big]. \tag{2.9}$$

We integrate by parts in (2.9). If the sum of the integrals in the angle brackets in (2.9) is not equal to zero, then the upper boundary of (2.8) is equal to $+\infty$. In that case, (2.7) has a minimum which is not equal to $+\infty$ if

$$p^*_{ij,j} = 0 \ \text{ in } P_1, \quad p^*_{ij} n_j = 0 \ \text{ on } \gamma \cup \Gamma, \tag{2.10}$$

which corresponds to the equilibrium equation and boundary conditions in stresses.

Allowing for the fact that the values of the function $\mathbf{u} \in V$ coincide at the boundaries ω_+ and ω_-, we can write the last integral in (2.9) in the form

$$\int_{\omega_+} [p^*_{ij}] n_j u_i\, dy, \tag{2.11}$$

where [ ] denotes the difference in the values of the function at the boundaries of the periodicity cell ω_+ and ω_- (Fig. 1). In deriving (2.11), we considered that the normals to these boundaries oppose one another. The periodicity condition for $\mathbf{u} \in V$ must be satisfied in order to have integral (2.11) vanish for any $p^*_{ij} n_j$.

We now introduce the set of allowable stresses

$$\Sigma = \{\sigma_{ij} \in H: \sigma_{ij,j} = 0 \text{ in } P_1,\ \sigma_{ij} n_j = 0 \text{ on } \gamma\cup\Gamma,\ \sigma_{ij}(\mathbf{y}) \text{ is periodic over } y_m\ (m \in D)\}. \tag{2.12}$$

We identify the space H with its conjugate space H^* (σ_{ij} is identified with p_{ij}). Then Eq. (2.8) takes the form

$$h^*(p^*) = -G^*(\sigma) + \chi_\Sigma(\sigma), \tag{2.13}$$

where χ_Σ is the indicator function of the set Σ [$\chi_\Sigma(\sigma) = 0$ if $\sigma_{ij} \in \Sigma$ and $\chi_\Sigma(\sigma) = +\infty$ if $\sigma_{ij} \notin \Sigma$]. We introduce the Castigliano functional $J_\sigma(\sigma) = -G^*(\sigma)$. Then, with allowance for (2.13) problem (2.7) takes the form

$$J_\sigma(\sigma) \to \min, \quad \sigma_{ij} \in \Sigma. \tag{2.14}$$

The Korn inequality [1-6] is valid for the functions of V_0. Then the conditions of theorem III [8, Sec. 4.1] are satisfied on V_0, so that we obtain the equality $\max\limits_{\mathbf{u}\in V_0} J_u(\mathbf{u}) = \min\limits_{p^*\in H^*} h^*(p^*)$. Since $J_u(\mathbf{u})$ is independent of the terms that correspond to the displacement of a rigid body, we have the equality

$$\max_{\mathbf{u}\in V} J_u(\mathbf{u}) = \min_{p^*\in H^*} h^*(p^*),$$

which in the present case takes the form

$$\max_{\mathbf{u}\in V} J_u(\mathbf{u}) = \min_{\sigma_{ij}\in\Sigma} J_\sigma(\sigma). \tag{2.15}$$

Using Eq. (2.6), we obtain the following equality from (2.15):

$$A^M = \langle a_{ijkl}(\mathbf{y}) f^M_{ij}(\mathbf{y}) f^M_{kl}(\mathbf{y})\rangle - 2\max_{\mathbf{u}\in V} J_u(\mathbf{u}), \tag{2.16}$$

$$A^M = \langle a_{ijkl}(\mathbf{y}) f^M_{ij}(\mathbf{y}) f^M_{kl}(\mathbf{y})\rangle - 2\min_{\sigma_{ij}\in\Sigma} J_\sigma(\sigma),$$

which represents two variational principles (in displacements and in stresses) for rigidity.

With arbitrary $\mathbf{u} \in V$, $\sigma_{ij} \in \Sigma$, we use (2.16) to obtain a two-sided estimate

$$\langle a_{ijkl}(\mathbf{y}) f^M_{ij}(\mathbf{y}) f^M_{kl}(\mathbf{y})\rangle - 2J_u(\mathbf{u}) \geqslant A^M \geqslant \langle a_{ijkl}(\mathbf{y}) f^M_{ij}(\mathbf{y}) f^M_{kl}(\mathbf{y})\rangle - 2J_\sigma(\sigma). \tag{2.17}$$

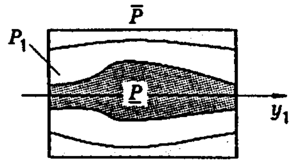

Fig. 2. Periodicity cells: P_1 is the initial cell, $\overline{P}$ is the circumscribed cell, and $\underline{P}$ is the inscribed cell.

The functional $G^*(p^*)$ is easily calculated [8]:

$$G^*(p^*) = -\frac{1}{2} a^{-1}_{ijkl}(\mathrm{y}) p^*_{ij} p^*_{kl} - f^M_{ij}(\mathrm{y}) p^*_{ij} - \frac{1}{2} a_{ijkl}(\mathrm{y}) f^M_{ij}(\mathrm{y}) f^M_{kl}(\mathrm{y}). \tag{2.18}$$

Here a^{-1}_{ijkl} is the tensor that is the inverse of a_{ijkl}.

3. Estimates for Rigidities. Inscribed and Circumscribed Periodicity Cells. For materials with voids, we can introduce inscribed $\underline{P}$ and circumscribed $\overline{P}$ periodicity cells into the discussion: $\underline{P} \subset P_1 \subset \overline{P}$. Here the boundaries of the periodicity cell $\underline{P}$ and $\overline{P}$, which intersect ω_+ and ω_-, must be congruent (Fig. 2).

The variational principle in displacements (2.16) can be written in the form

$$A^M = \min_{\mathbf{u} \in V} f(\mathbf{u}), \tag{3.1}$$

where

$$F(\mathbf{u}) = \langle a_{ijkl}(\mathrm{y})(u_{i,j} + f^M_{ij}(\mathrm{y}))(u_{k,l} + F^M_{kl}(\mathrm{y}))\rangle.$$

Equation (3.1) follows from (2.1)–(2.3) with allowance for the known symmetries of the elastic constants a_{ijkl} [7].

Now we introduce the set of possible displacements $\underline{V}$ and $\overline{V}$ by replacing P_1 in (2.4) by $\underline{P}$ and $\overline{P}$, respectively. We also define three functionals $\underline{F}(\mathbf{u})$, $F(\mathbf{u})$, and $\overline{F}(\mathbf{u})$ on one space $\overline{V}$, assuming that $a_{ijkl}(\mathrm{y}) = 0$ outside $\underline{V}$ for the functional $\underline{F}(\mathbf{u})$ and that $a_{ijkl}(\mathrm{y}) = 0$ outside P_1 for $F(\mathbf{u})$. By virtue of conditions (1.6) and (1.7), for any $\mathbf{u} \in V$ we have

$$\underline{F}(\mathbf{u}) \leqslant F(\mathbf{u}) \leqslant \overline{F}(\mathbf{u}); \tag{3.2}$$

$$\min_{\mathbf{u} \in \overline{V}} \underline{F}(\mathbf{u}) = \min_{\mathbf{u} \in V} \underline{F}(\mathbf{u}), \quad \min_{\mathbf{u} \in \overline{V}} F(\mathbf{u}) = \min_{\mathbf{u} \in V} F(\mathbf{u}). \tag{3.3}$$

Having taken the minimum in (3.2) for $\mathbf{u} \in \overline{V}$ and using (3.3) and the variational principle (3.1), which is valid for each $\underline{P}$, P_1, and $\overline{P}$ from the periodicity cell, we obtain

$$\underline{A}^M \leqslant A^M \leqslant \overline{A}^M, \tag{3.4}$$

where $\underline{A}^M$ and $\overline{A}^M$ are the rigidities of the structures formed on the basis of the inscribed and circumscribed periodicity cell.

Formula (3.4) generalizes a well-known fact of the mechanics of finite-dimensional structures: the addition or removal of a constraint does not decrease or increase rigidity.

Now we examine the case where a periodicity cell P_1 is divided into two nonintersecting periodicity cells P^1 and P^2, the projections S of which coincide with the projection of P_1. We use the variational principle in stresses. We introduce the set of allowable stresses Σ^1 and Σ^2 by replacing P_1 by P^1 and P^2 in (2.12). For the allowable stresses determined on P^1, we introduce the continuation operator

$$C\sigma_{ij} = \begin{cases} \sigma_{ij}(\mathrm{y}) & \text{for } \mathrm{y} \in P^1, \\ 0 & \text{for } \mathrm{y} \in P_1 \backslash P^1. \end{cases} \tag{3.5}$$

We prove that $C\sigma_{ij} \in \Sigma$. To do this, it is sufficient to establish that $\sigma^c_{ij} = C\sigma_{ij}$ satisfies the equality

$\sigma^c_{ij,j} = 0$. We multiply $\sigma^c_{ij}\ \varphi_{,j}$, where $\varphi(\mathrm{y})$ belongs to the set of smooth finite functions $\mathcal{D}(P_1)$ [9]. Integrating by parts with allowance for the fact that $\sigma_{ij}n_j = 0$ on γ, we obtain the equality

$$\int_{P_1} \sigma^c_{ij}\varphi_{,j}\, d\mathrm{y} = -\int_{P_1} \sigma_{ij,j}\varphi\, d\mathrm{y} + \int_{\gamma} \sigma_{ij}n_j\varphi_i\, d\mathrm{y} \tag{3.6}$$

for any $\varphi \in \mathcal{D}(P_1)$, which proves the above proposition.

Similarly to (3.5), we introduce the continuation operator from Σ^2 into Σ. We use for it the same notation C.

Similarly to (3.6), it can be proved that $C\sigma^1_{ij} + C\sigma^2_{ij} \in \Sigma$ for any $\sigma^1_{ij} \in \Sigma^1$ and $\sigma^2_{ij} \in \Sigma^2$.

Since $C\Sigma^1 + C\Sigma^2 \subset \Sigma$ ($C\Sigma^i$ denotes the image of Σ^i in the continuation of C), we have

$$\max_{\sigma_{ij}\in\Sigma} J_\sigma(\sigma) \geqslant \max_{\substack{\sigma^1_{ij}\in\Sigma^1\\ \sigma^2_{ij}\in\Sigma^2}} J_\sigma(C\sigma^1_{ij} + C\sigma^2_{ij}) = \max_{\sigma^1_{ij}\in\Sigma^1} J^1(\sigma^1) + \max_{\sigma^2_{ij}\in\Sigma^2} J^2(\sigma^2), \tag{3.7}$$

where J^1 and J^2 is the contraction of the function J_σ on Σ^1 and Σ^2 (P^1 and P^2 are the regions of integration). We represent the functional J_σ on the set $C\Sigma^1 + C\Sigma^2$ in the form of a sum by virtue of definition (2.18) and the fact that σ^1_{ij} and σ^2_{ij} have nonintersecting carriers (the regions in which they are nontrivial). We also represent the functional $\langle a_{ijkl}(\mathrm{y})f^M_{ij}(\mathrm{y})f^M_{kl}(\mathrm{y})\rangle$ in the form of a sum.

In accordance with (3.7) and the variational principle in stresses (2.16), we obtain

$$\begin{aligned} A^M &= \langle a_{ijkl}(\mathrm{y})f^M_{ij}(\mathrm{y})f^M_{kl}(\mathrm{y})\rangle - 2\max_{\sigma_{ij}\in\Sigma} J_\sigma(\sigma) \leqslant \langle a_{ijkl}(\mathrm{y})f^M_{ij}(\mathrm{y})f^M_{kl}(\mathrm{y})\rangle_1 \\ &- 2\max_{\sigma^1_{ij}\in\Sigma^1} J^1(\sigma^1) + \langle a_{ijkl}(\mathrm{y})f^M_{ij}(\mathrm{y})f^M_{kl}(\mathrm{y})\rangle_2 - 2\max_{\sigma^2_{ij}\in\Sigma^2} J^2(\sigma^2), \end{aligned} \tag{3.8}$$

where $\langle\ \rangle_1$ and $\langle\ \rangle_2$ are the means of P^1 and P^2 over the periodicity cell.

As a result of applying the variational principle (2.16) to periodicity cells P^1 and P^2, we obtain the following inequality from (3.8):

$$A^M \geqslant A^1 + A^2. \tag{3.9}$$

Having written inequality (3.9) for the inscribed and circumscribed cells, we obtain

$$\overline{A}^M \leqslant A^M + \overline{\delta}, \quad \underline{A}^M \leqslant A^M - \underline{\delta}, \tag{3.10}$$

where $\overline{\delta}$ and $\underline{\delta}$ are calculated from the formula $\langle a_{ijkl}(\mathrm{y})f^M_{ij}(\mathrm{y})f^M_{kl}(\mathrm{y})\rangle_\Delta - 2\max\limits_{\sigma_{ij}\in\Sigma(\Delta)} J_\sigma(\sigma)$, in which $\Delta = \overline{P}\backslash P_1$ for $\overline{\delta}$ and $\Delta = P_1\backslash\underline{P}$ for $\underline{\delta}$.

Here $\underline{\delta}(\overline{\delta})$ is the rigidity of the structure Δ, equal to the difference between P_1 and the inscribed (circumscribed) structures.

It follows from (3.10) that the rigidity of the composite structure is less than the sum of the rigidities of its components.

Voigt Estimate (Estimate from Above). We take the allowable displacements in the form $\mathbf{u} = 0$. We find from (2.17) that

$$A^M \leqslant \langle a_{ijkl}(\mathrm{y})f^M_{ij}(\mathrm{y})f^M_{kl}(\mathrm{y})\rangle. \tag{3.11}$$

Reiss Estimate (estimate from below). Let y_m ($m \in D$) be the variables over which periodicity occurs. Let y_f be an additional variable (which does not exist for three-dimensional composites; this is y_3 for plates and y_2 or y_3 for beams). We assume that the periodicity cell is planar (for the plate) or cylindrical (for the beam). Then the allowable stresses have the form

$$\sigma_{ij} = C_{ij}y_f^n \tag{3.12}$$

(C_{ij} are arbitrary constants for the three-dimensional composite; $C_{i3} = C_{3i} = 0$ for the plate and $C_{11} \neq 0$ only for the beam; n is an arbitrary integer for plates and beams, while $n = 0$ for three-dimensional composites).

With allowance for (2.18), the estimate of (2.17) from below can be written in the form

$$A^M \geqslant \langle -a^{-1}_{ijkl}(\mathrm{y})\sigma_{ij}\sigma_{kl} + 2f^M_{kl}(\mathrm{y})\sigma_{kl}\rangle. \tag{3.13}$$

With allowance for (3.12), we write the right side of inequality (3.13) as follows:

$$-C_{ij}C_{kl}\langle a^{-1}_{ijkl}(\mathrm{y})y_f^{2n}\rangle + C_{kl}\langle f^M_{kl}(\mathrm{y})y_f^n\rangle. \tag{3.14}$$

There are no restrictions on C_{ij}. We subject (3.14) to unconditional maximization with respect to C_{ij}. The Euler equation for (3.14) has the form

$$-C_{kl}\langle a^{-1}_{ijkl}(\mathrm{y})y_f^{2n}\rangle + \langle f^M_{ij}(\mathrm{y})y_f^n\rangle = 0.$$

Its solution is

$$C_{ij} = \langle a^{-1}_{ijkl}(\mathrm{y})y_f^{2n}\rangle^{-1}\langle f^M_{kl}(\mathrm{y})y_f^n\rangle \tag{3.15}$$

(the superscript -1 denotes inversion of the tensor of the corresponding dimensionality).

Substitution of (3.15) into (3.14) gives

$$A^M \geqslant \langle a^{-1}_{ijkl}(\mathrm{y})y_f^{2n}\rangle^{-1}\langle f^M_{ij}(\mathrm{y})y_f^n\rangle\langle f^M_{kl}(\mathrm{y})y_f^n\rangle. \tag{3.16}$$

4. Examples of Estimates. Let us give the form of the function $f^M_{ij}(\mathrm{y})$ and values of the multiple superscript M that characterize concrete structures — three-dimensional composites, plates, and beams.

For a three-dimensional composite [1–3, 9], we have

$$f^M_{ij}(\mathrm{y}) = \delta_{ik}\delta_{jl}, \quad M = (ijij),\ i, j = 1, 2, 3.$$

Using these formulas and (3.10) and (3.16), we arrive at the classical Lagrange and Castigliano functionals and the Voigt and Reiss estimates for the averaged elastic constants A_{ijij}

$$\langle a_{ijij}(\mathrm{y})\rangle \geqslant A_{ijij} \geqslant \langle a^{-1}_{ijij}(\mathrm{y})\rangle^{-1}. \tag{4.1}$$

An estimate for the averaged coefficients of elliptic equations, which is similar to (4.1), was obtained in [10].

The functions f^M_{ij} for a plate have the form [3, 4, 9]

$f^{\alpha\beta}_{ij}(\mathrm{y}) = \delta_{i\alpha}\delta_{j\beta}$, $M = (\alpha\beta\alpha\beta 0)$ for tensile–compressive rigidities and

$$f^{\alpha\beta}_{ij}(\mathrm{y}) = -\delta_{i\alpha}\delta_{j\beta}y_3, \quad M = (\alpha\beta\alpha\beta 2), \quad \alpha, \beta = 1, 2; \quad i, j = 1, 2, 3$$

for bending–torsional rigidities.

Using these formulas, we arrive at the following estimates for the rigidities of the plate:

$\langle a_{\alpha\beta\alpha\beta}(\mathrm{y})\rangle \geqslant A^0_{\alpha\beta\alpha\beta} \geqslant \langle a^{-1}_{\alpha\beta\alpha\beta}(\mathrm{y})\rangle^{-1}$ in tension–compression and

$\langle a_{\alpha\beta\alpha\beta}(\mathrm{y})y_3^2\rangle \geqslant A^2_{\alpha\beta\alpha\beta} \geqslant \langle y_3^2\rangle^2\langle a^{-1}_{\alpha\beta\alpha\beta}(\mathrm{y})y_3^2\rangle^{-1}$ in bending/torsion.

The functions f^M_{ij} for a beam have the form [9]

$f^M_{ij}(\mathrm{y}) = \delta_{i1}\delta_{j1}$, $M = (0)$ in tension/compression,

$f^M_{ij}(\mathrm{y}) = -\delta_{i1}\delta_{j1}y_\alpha$, $M = (\alpha)$ in bending, and

$f^M_{ij}(\mathrm{y}) = S_\gamma y_{\tilde\gamma}\delta_{i1}\delta_{j1}$, $M = (b)$ ($i, j = 1, 2, 3$, $\gamma = 2, 3$, $\tilde\gamma = 2$ if $\gamma = 3$, $\tilde\gamma = 3$ if $\gamma = 2$, $S_2 = 1$, and $S_3 = -1$) in torsion.

Using these formulas, we obtain two-sided estimates for the rigidities of the beam:

$\langle a_{1111}(\mathrm{y})\rangle \geqslant A^0 \geqslant 1/\langle 1/a_{1111}(\mathrm{y})\rangle$ in tension,

$\langle a_{1111}(\mathrm{y})y_\alpha^2\rangle \geqslant A^\alpha \geqslant \langle y_\alpha^2\rangle^2/\langle y_\alpha^2/a_{1111}(\mathrm{y})\rangle$ in bending, and

$\langle a_{1\gamma 1\gamma}(\mathrm{y})y_{\tilde\gamma}^2\rangle \geqslant A^b \geqslant \langle y_2^2 + y_3^2\rangle^2/\langle (y_2^2 + y_3^2)/a_{1\gamma 1\gamma}(\mathrm{y})\rangle$ in torsion.

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